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LETTER TO THE EDITOR

On the Hamiltonian structure of 2D ODE possessing an invariant

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Abstract. We discuss the Hamiltonian structure with respect to the canonical bracket for a 2D dynamical system and prove that this structure can be obtained simply with a rescaling of the independent variable. The rescaling will provide a Hamiltonian when a time-independent invariant is known, the Hamiltonian coinciding with the invariant. Applications to different systems of ODE in two dimensions are presented.

In two recent papers Nutku [1, 2] succeeded in finding the Hamiltonian structure of several dynamical systems for which the naive criterion for the existence of Hamiltonian structure fails. The condition for such a possibility was the existence of invariants of the motion (one or two if the dimension of the systems is two or three respectively). Then he was able to associate structure functions [3] to each dynamical system he considered, these functions allowing him to cast the systems in a Hamiltonian structure with respect to a generalized Poisson bracket. Here we are concerned with the Hamiltonian structure with respect to the canonical bracket and prove that this structure can be obtained simply with a rescaling of the independent variable (the time). The rescaling will provide a Hamiltonian to any two-dimensional dynamical system for which a time-independent invariant is known and, as expected, we find that, then, the Hamiltonian is simply the invariant. Several applications are applied to different systems of 2D Lotka–Volterra (LV) equations for which one time-independent invariant is known [4]. The theory is extended to the case of a time-dependent invariant which can be rescaled to exhibit a time-independent form. The letter is organized as follows. It starts by presenting the general results concerning the systems of ODE followed by three applications to LV possessing invariants. One of these concerns the 2D LV system studied by Nutku for which we deduce the Hamiltonian formalism, and investigate the period while the others concern more general LV systems for which invariants can be found.

We begin by discussing a general result on systems of ODE.

Theorem. Any two-dimensional system of ODE possessing a time-independent invariant can be written in a Hamiltonian form and the Hamiltonian is the invariant.

Proof. Let us consider a dynamical system

$$\frac{dx}{dt}=f(x,y,t) \qquad \frac{dy}{dt}=g(x,y,t). \tag{1}$$

Assume the existence of a time-independent first integral $I(x,y)$ and consequently write

$$dI=\frac{\partial I}{\partial x}dx+\frac{\partial I}{\partial y}dy=0. \tag{2}$$

Introducing a new time θ one obtains

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=-\frac{\partial I}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{1}{\partial I/\partial y}.$$

If we define $\mathrm{d}x/\mathrm{d}\theta=\partial I/\partial y$ then we have $\mathrm{d}y/\mathrm{d}\theta=-\partial I/\partial x$ and the system of these equations constitutes the Hamiltonian form of system (1) with $H(x,y)=I(x,y)$. In fact $\mathrm{d}x/\mathrm{d}\theta=\partial I/\partial y$ defines the new time element $\mathrm{d}\theta$. One can write it as

$$\mathrm{d}\theta=J\,\mathrm{d}t=\frac{f(x,y,t)}{\partial I/\partial y}\mathrm{d}t \tag{3}$$

and as y is a function of x and I, the relation between θ and t is reduced to a quadrature. One can similarly 'label' one trajectory I with $\mathrm{d}\theta=\mathrm{d}x/(\partial I/\partial y)$.

Note that in the original time the system can be written with (3) as follows

$$\frac{\mathrm{d}x}{\mathrm{d}t}=J\frac{\partial H}{\partial y}\qquad\frac{\mathrm{d}y}{\mathrm{d}t}=-J\frac{\partial H}{\partial x}. \tag{4}$$

Following Olver [3], system (4) can be considered as Hamiltonian with respect to a generalized Poisson bracket, J playing the role of the structure function. The bracket is defined as $\{FH\}=\nabla F\cdot \boldsymbol{v}_H$ where F is a smooth real-valued function and $\boldsymbol{v}_H=\boldsymbol{J}\cdot\nabla H$ is the vector field with the structure matrix

$$\boldsymbol{J}=\begin{vmatrix}0 & J\\ -J & 0\end{vmatrix}.$$

The introduction of the new time θ avoids the use of this structured form of the Hamiltonian system as it produces directly the canonical form.

Let us apply the preceding results to the first case reported by Nutku [1] of the LV system where the self-interaction terms are zero, namely

$$\frac{\mathrm{d}x}{\mathrm{d}t}=(a+by)x\qquad\frac{\mathrm{d}y}{\mathrm{d}t}=(a'+b'x)y \tag{5}$$

with $a, b'>0$ and $a', b<0$. Without any condition among these parameters, this system admits the well known integral (see for instance [5])

$$I=a\ln y+by-a'\ln x-b'x.$$

Now

$$\frac{\partial I}{\partial y}=\frac{a+by}{y}\qquad\frac{\partial I}{\partial x}=-\frac{a'+b'x}{x} \tag{6}$$

$$J=xy\qquad \mathrm{d}\theta=xy\,\mathrm{d}t. \tag{7}$$

With this value for J and (4) we obtain the Nutku's result. Here and in terms of the new time θ we deduce

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\tilde{f}\qquad\frac{\mathrm{d}y}{\mathrm{d}\theta}=\tilde{g} \tag{8}$$

with

$$\tilde{f}=\frac{a+by}{y}\qquad\tilde{g}=\frac{a'+b'x}{x} \tag{9}$$

where we remark that $\tilde{f}$ is a function only of y and $\tilde{g}$ of x. The consequence, on the evolution of an element, of the phase space is straightforward. In fact it is well known that the phase space volume element Δ satisfies the differential equation

$$\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t}=\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}. \tag{10}$$

Taking into account (5), we rewrite (10) as

$$\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t}=\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{11}$$

In the original time coordinate t, there is no conservation of the phase space volume but, instead, we obtain its periodic evolution, the maxima and minima of which happen when the second member of (11) cancels. Actually, these extremals are located on the line

$$a+by+a'+b'x=0$$

provided of course, that the volume element is sufficiently small in order that the relation (10) be valid. A computation showing this property appears in figure 1.

In the new time the equations of evolution of Δ read:

$$\delta\Delta=\Delta\left(\frac{\partial\tilde{f}}{\partial x}+\frac{\partial\tilde{g}}{\partial y}\right)\delta\theta. \tag{12}$$

As $\tilde{f}$ is independent of x and $\tilde{g}$ is independent of y, we immediately find the well known conservation of the volume element $\delta\Delta=0$.

Note that the time θ is now dependent on the trajectories since $\mathrm{d}\theta=xy\,\mathrm{d}t$ and the phase space volume conservation is obtained by selecting the systems at different times t. From a mathematical point of view θ and t play the same role while from a physical point of view we like to come back to the original t.

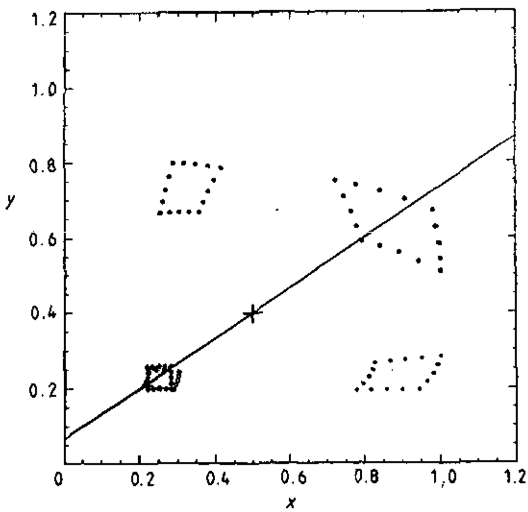

Figure 1. Evolution of a volume element in phase space: equation (5) with $a=1.2$, $a'=-1$, $b=-3$, $b'=2$.

Let us integrate over one period the equations (5) multiplied by $\mathrm{d}t$:

$$a\oint x\,\mathrm{d}t+b\oint xy\,\mathrm{d}t=0 \qquad a'\oint y\,\mathrm{d}t+b'\oint xy\,\mathrm{d}t=0. \tag{13}$$

Recalling from (7) that the new time is $\mathrm{d}\theta=xy\,\mathrm{d}t$, one can define the new period as

$$\Theta=\oint xy\,\mathrm{d}t. \tag{14}$$

Let us rewrite equations (5) in the form

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}=a+by \qquad \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}=a'+b'x.$$

Then multiply by $\mathrm{d}t$ and integrate over one period. We see immediately that the first members cancel and if T is the period in the old time, we have

$$aT+b\oint y\,\mathrm{d}t=0 \qquad a'T+b'\oint x\,\mathrm{d}t=0. \tag{15}$$

Eliminating $\oint x\,\mathrm{d}t$ and $\oint y\,\mathrm{d}t$ between (15) and (13) and using (14) results in

$$bb'\Theta=aa'T.$$

It is interesting to remark that Θ and T are proportional. It is well known that if x_0 and y_0 (the initial values) $\to 0$, then T goes to infinity and consequently so does Θ. But while, in the old system, this was due to the slowness of the motion near the origin, in the new system this slowness disappears since $\mathrm{d}\theta=xy\,\mathrm{d}t$ but now it is far from the origin that the slowness appears and altogether the two effects cancel within a constant factor.

The more general two species LV system, under invariant conditions type III, can be written in the following form with the essential parameters a, a', R, R':

$$\frac{\mathrm{d}x}{\mathrm{d}t}=ax(1+x+Ry) \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=a'y(1+R'x+y). \tag{16}$$

This system admits a first integral [4]:

$$I=x^{\alpha}y^{\beta}(1+x+y) \tag{17}$$

with the following values for α and β

$$\alpha=\frac{a-R'a'}{a(RR'-1)} \qquad \beta=\frac{a'-Ra}{a'(RR'-1)}. \tag{18}$$

An existence condition for this first integral is given by the following relation

$$Ra+R'a'=a+a'. \tag{19}$$

The preceding condition implies

$$\alpha+1=R'\alpha \qquad \beta+1=R\beta. \tag{20}$$

Using this property in the computation of the partial derivatives of I over x and y, the LV system takes the form (8) with

$$\tilde{f}=\beta x^{\alpha}y^{\beta-1}(1+x+Ry) \qquad \tilde{g}=-\alpha x^{\alpha-1}y^{\beta}(1+R'x+y). \tag{21}$$

Finally the relation between the 'old' and the 'new' time now reads:

$$\mathrm{d}\theta = \frac{aa'(RR'-1)}{a'-Ra}\frac{\mathrm{d}t}{x^{\alpha-1}y^{\beta-1}}.$$

Again the relation between θ and t depends on the trajectory (i.e. on the value of the invariant). To a time θ corresponds different times t and it is this introduction of a time attached to the trajectory that is responsible for the conservation of the phase space volume. Note that the case where $a=-a'=1$, $R=R'=2$, which from (20) corresponds to $\alpha=\beta=1$, is the case where (16) has the Hamiltonian form in the original time t.

As before, in the original coordinates, the evolution of the phase space volume is computed straightforwardly as follows

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y} = ax+a'y+a(1+x+Ry)+a'(1+R'x+y)$$

$$= ax+a'y+\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t}$$

using (10). One obtains

$$\Delta = \Delta_0 \exp \oint (ax+a'y)\,\mathrm{d}t. \tag{22}$$

In the simplified LV system (5), we have seen that Δ is periodic and comes back to its original value after a period. Let us show that this is also true in the general case. For that, rewrite equations (16) in the form

$$\frac{1}{ax}\frac{\mathrm{d}x}{\mathrm{d}t}=1+x+Ry \qquad \frac{1}{a'y}\frac{\mathrm{d}y}{\mathrm{d}t}=1+R'x+y.$$

Let us multiply by $\mathrm{d}t$ and integrate on a cycle. Let us call T the period and X and Y the corresponding integrals $\int x\,\mathrm{d}t$ and $\oint y\,\mathrm{d}t$. Now if we remark that the first members are exact differentials we have

$$T+X+RY=0 \qquad T+Y+R'X=0$$

from which we obtain

$$X=T\frac{R-1}{1-RR'} \qquad Y=T\frac{R'-1}{1-RR'}$$

and consequently

$$aX+a'Y=0 \tag{23}$$

taking into account (19). Now from (22), the infinitesimal volume recovers its initial value after a cycle. As before, the property cannot be extended to finite volumes since the different points do not have the same period. Contrarily in the rescaled space there is conservation of the phase space volume, as expected from a Hamiltonian system: the points do not always have the same period but the volume element is conserved at any time. One can proceed without difficulty from the conservation of the infinitesimal volume to a finite volume.

For our next application we consider an LV system with first integral conditions of type II. When the condition is

$$a = a' \tag{24}$$

then the LV system (16) admits the following time-dependent invariant [4]:

$$I = x^{\alpha} y^{\beta} (Ax + By)\,\mathrm{e}^{st} \tag{25}$$

where α and β are given by (18) with $a = a'$ and

$$A = R' - 1 \qquad B = 1 - R \qquad s = -a(1 + \alpha + \beta). \tag{26}$$

Now, with the following rescaling

$$\tilde{x} = x\,\mathrm{e}^{-at} \qquad \tilde{y} = y\,\mathrm{e}^{-at}$$

the time-dependence in (25) disappears using the definition (26) of s. The invariant now reads

$$I = \tilde{x}^{\alpha} \tilde{y}^{\beta} (A\tilde{x} + B\tilde{y})$$

and the LV system becomes

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = a\tilde{x}(\tilde{x} + R\tilde{y})\,\mathrm{e}^{at} \qquad \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = a\tilde{y}(R'\tilde{x} + \tilde{y})\,\mathrm{e}^{at}.$$

Now the Hamiltonian form is obtained with

$$\begin{aligned} \tilde{f} &= \frac{\partial I}{\partial \tilde{y}} = \tilde{x}^{\alpha} \tilde{y}^{\beta - 1} [A\beta\tilde{x} + B(\beta + 1)\tilde{y}]\,\mathrm{e}^{at} \\ \tilde{g} &= -\frac{\partial I}{\partial \tilde{x}} = \tilde{x}^{\alpha - 1} \tilde{y}^{\beta} [A(\alpha + 1)\tilde{x} + B\alpha\tilde{y}]\,\mathrm{e}^{at} \end{aligned} \tag{27}$$

and the relation between the 'old' and the 'new' time from (3) and (4) is

$$\mathrm{d}\theta = a\tilde{x}^{1-\alpha}\tilde{y}^{1-\beta} \frac{\tilde{x} + R\tilde{y}}{A\beta\tilde{x} + B(\beta + 1)\tilde{y}} \exp(at)\,\mathrm{d}t.$$

We have found the particular role of a time-independent integral of a two-dimensional dynamical system; it allows us to transform the system into a Hamiltonian one. We note that only the timescale must change for this purpose. The examples are on the LV equations and the main results are the following. Concerning the phase space volume, in the original space the volume fluctuates on a cycle passing through maxima and minima. For the systems having time-dependent invariants, one can try a rescaling of the dependent variables in order to transform the invariant in a time-independent form, as in the case of the LV system when it models a competitive process. Although not done here, one can guess a Hamiltonian character when two invertible invariants are known for a system of three equations. This is due to the fact that the existence of an invariant reduces the complexity of a dynamical system by one unit, the result of which can be generalized.

References

[1] Nutku Y 1990 Hamiltonian structure of the Lotka–Volterra equations *Phys. Lett.* **145A** 27

[2] Nutku Y 1990 Bi-Hamiltonian structure of the Kermack–McKendrick model for epidemics *J. Phys. A: Math. Gen.* **23** L1145

[3] Olver P J 1986 *Applications of Lie Groups to Differential Equations (Graduate Texts in Mathematics 107)* (New York: Springer)

[4] Cairó L and Feix M R 1992 Families of invariants of the motion for the Lotka–Volterra equations: The linear polynomials family *J. Math. Phys.* **33** 2440

[5] Davis H T 1960 *Introduction to Nonlinear Differential Equations* (New York: Dover)